4.2 Mechanical System Characteristics

The following concerns the characteristics usually needed for vibration analysis:

MASS - this is weight divided by the gravitational constant ($g = 386$ in/sec^2). Determine by finding weight of part. Either by direct weighing, or by computation of product of volume and density.

CENTER OF GRAVITY - Can be obtained by computation or experiment. The experiment is suggested by the definition of the C.G. as the point of support at which the body will be in equilibrium. For example, a plane body, or one of constant thickness, can be supported on a peg; when in equilibrium, a vertical line drawn through the peg will pass through the center of gravity. If this experiment is repeated with a different peg location relative to the body, the center of gravity will be the point of intersection of two lines. Similar experiments, though somewhat more difficult to devise, can be conducted for three-dimensional bodies.

MOMENTS OF INERTIA - For standard shapes these are tabulated in the handbooks. A few of the more commonly used shapes are tabulated below. To determine the mass moment of inertia of a body, the cross-sectional area of which is constant, multiply the area moment of inertia by the product of the length of the body and its density [mass density = (lbs/in^3)/g]. Consider, for example, a rectangular, steel bar, 10" long, 2" wide and 2" high, as shown in Figure 8.

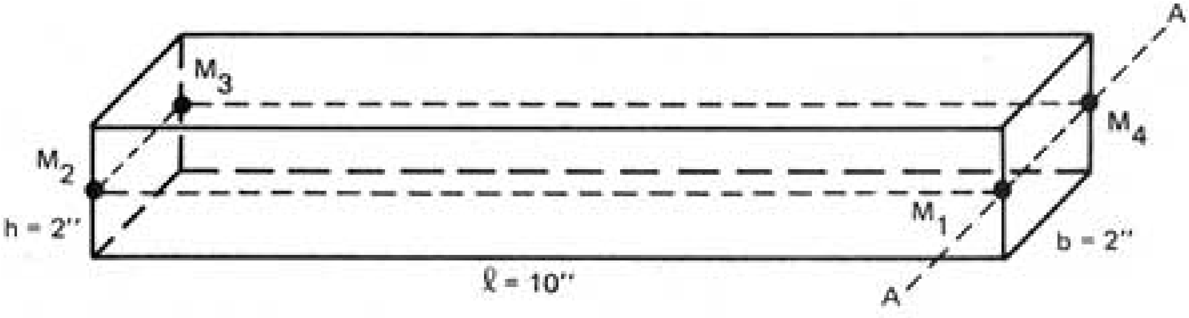
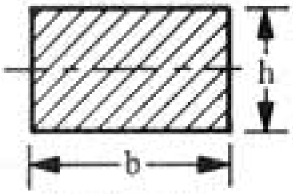
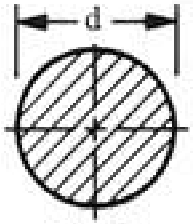
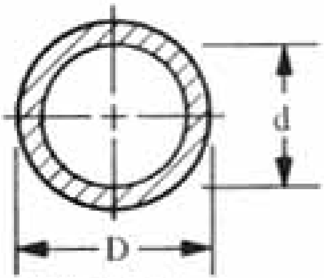


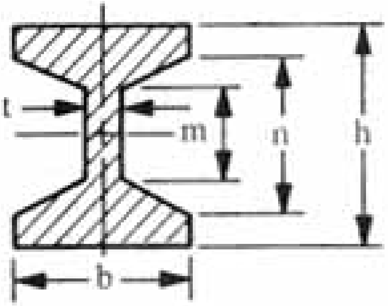
Figure 8

Cross-section	*Area moment of inertia about axis indicated; if the linear dimensions are in inches, the units of area moment of inertia are (inches)^4.*
 (Rectangle)	$\frac{bh^3}{12}$ (For square, b = h)
 (Circle)	$\frac{\pi d^4}{64}$



(Tubular)

$$\frac{\pi}{64}(D^4 - d^4)$$



I - Beam

Horizontal axis:—

$$\frac{1}{12}\left[bh^3 - \frac{(b-t)(n^4-m^4)}{4(n-m)}\right]$$

Vertical axis:—

$$\frac{1}{12}\left[b^3(h-n) + mt^3 + \frac{(n-m)(b^4-t^4)}{4(b-t)}\right]$$

The area moment of inertia about axis AA is

$$\frac{bh^3}{12} \text{ or } \frac{(2)(2)^3}{12} = 1.333 \text{ in.}^4$$

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The mass moment of inertia of the bar about the midplane, $M_1M_2M_3M_4$ (containing axis AA)

$$= \text{Area Moment x Length of Bar x } \frac{\text{Mass Density}}{\text{Gravitational Constant}}$$

Assuming a value of 0.281 lbs/in^3 for the density of steel and a value of 386 in/sec^2 for the gravitational constant, the mass moment of inertia, in units of in-lb-sec^2, is given by

$$1.333 \text{ x } 10 \text{ x } \frac{0.281}{386}$$

$$\text{or } 0.00971 \text{ in-lb-sec}^2$$

The moment of inertia of complicated machine parts can be calculated or determined experimentally. Experimental setups usually involve a compound-pendulum experiment. The part (rotor, etc.) may be suspended by a knife edge or wire, etc. and permitted to swing about an axis, which is parallel to the axis about which the mass moment of inertia is desired. See Figure 9.

Let d = distance from center of gravity to point of support (knife edge, or end of wire, etc.)
T = period of pendulum vibration in seconds; (measure several and divide by their number).
W = weight of part, lbs
g = 386 in/sec^2

Then the moment of inertia, l, about an axis through the center of gravity parallel to the swing axis is given by

$$I = (W/g)\left(\frac{T^2dg}{4\pi^2} - d^2\right) \text{ in-lb-sec}^2$$

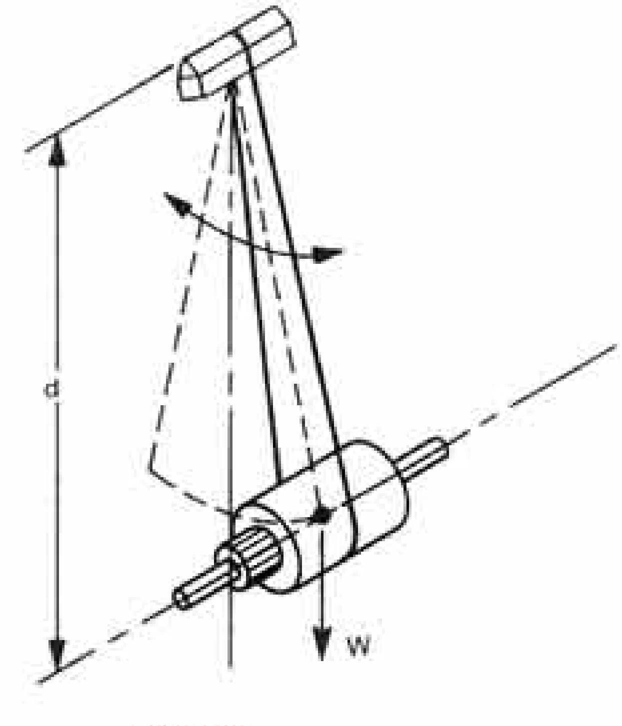


Figure 9

Similar measurements can be made with the part mounted so as to vibrate as a torsional pendulum (see W.l. Senger, Machine Design, Nov., Dec., 1944, Jan.-Feb., 1945).

Products of inertia are required more seldom than moments of inertia. Their experimental determination is more difficult. One way, usable in parts which function as rotors, is to mount the rotor on bearings and permit the rotor to rotate at speed. If the rotor unbalance is known, the bearing reactions are directly relatable to the products of inertia (see, for example, Housner and Hudson: "Applied Mechanics/Dynamics, Van Nostrand, 1959, p.224, Ex. 7.68).

If a moment of inertia about an axis through the center of gravity is known, the moment of inertia about a parallel axis, a distance D from the center of gravity is computed by the *Parallel-Axis Theorem*:

l(displaced axis) = l(about parallel axis through C.G.) + WD^2/g

where W is the weight of the body and g is the gravitational constant.

For geared systems, see below under elastic compliance.

ELASTIC COMPLIANCE (Spring Constants)

For mechanical springs, data in handbooks, etc. cover this subject fully. See, for example, "Mechanical Springs" by A.M. Wahl, Penton Publishing Co., 1944.

We give here only the main equations, which occur most frequently:

(a) Circular-Wire Helical Spring in Tension or Compression

$$k = \frac{d^4 G}{8D^3 N}$$

where k = spring constant lbs/in.
d = wire diameter of spring material, in.
D = mean coil diameter of spring (O.D.-d), in.
N = number of active turns of wire (usually total number less one, or one and one-half turns to allow for end effects).
G = shear modulus of spring material, lbs/in².

(b) Circular Wire Helical Spring in Torsion

$$k = \frac{Ed^4}{64DN} \text{ in-lbs/radian}$$

where the symbols are defined as in (a), above, and
E = elastic modulus (lbs/in²)

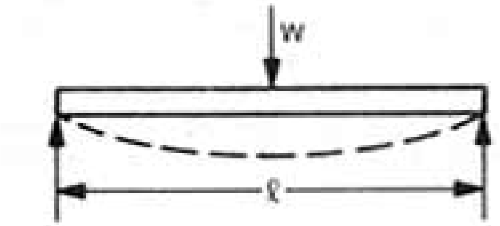
Springs also have lateral compliance, which is different from their axial compliance, see AM. Wahl above under "Elastic Compliance."

(c) Various Beam Configurations

For the most common forms of beams, the deflection formulae are as follows:

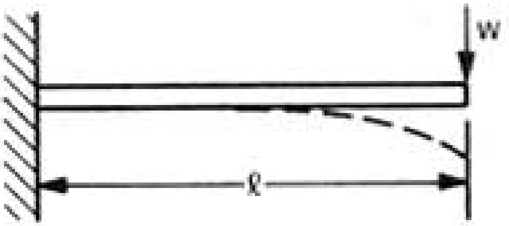
Spring constant, k, lbs/in (this is the weight, W, divided by the beam deflection at the weight).
E = elastic modulus of beam material, lbs/in2
l = area moment of inertia of beam cross-sectional area about neutral axis (axis through center of mass of cross-section parallel to the bending moment vector exerted by W)
ℓ = length of beam

Simply-supported beam; concentrated weight at middle.



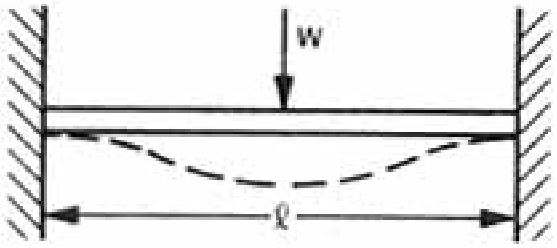
$$k = \frac{48EI}{\ell^3}$$

Cantilever beam; concentrated weight at end.



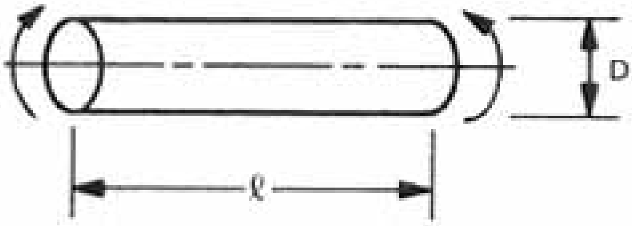
$$k = \frac{3EI}{\ell^3}$$

Beam with both ends built in; weight at middle.



$$k = \frac{192EI}{\ell^3}$$

(d) Torsional Stiffness of a Shaft (Bar in Torsion)

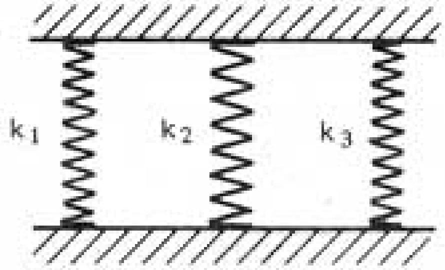


$$k = \frac{\pi D^4 G}{32\ell}$$

k = torsional stiffness, in-lbs per radian.
D = shaft diameter, in.
ℓ = length of shaft, in.
G = shear modulus of shaft material, lbs/in².

(e) Springs in Parallel

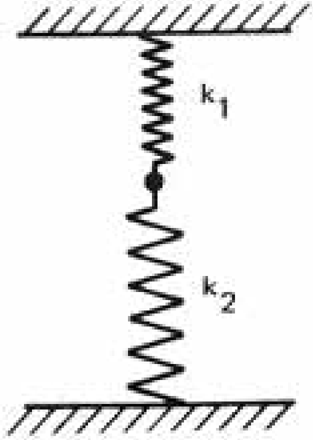
These combine like electrical resistances in series. This is the case when several springs support a single load, as shown. The springs are equivalent to a *single* spring, the spring constant of which is equal to the sum of the spring constants of the constituent springs. In the above sketch, the spring constant, k, of the single equivalent spring is given by:



$$k = k_1 + k_2 + k_3$$

(f) Springs in Series

These combine like electrical resistances in parallel. The equivalent single spring is weaker than any of the component springs. The spring constant, k, of the equivalent single spring is given by:



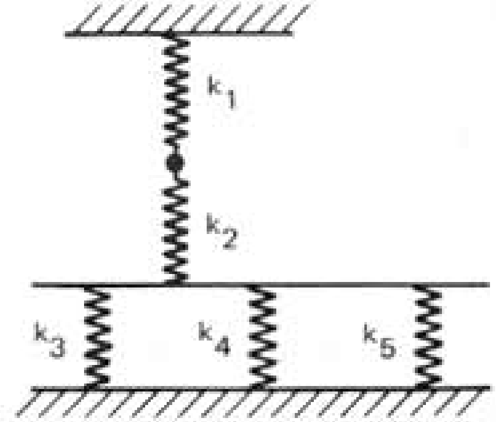
$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

If n springs are in parallel, this formula is readily extended:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \ldots\ldots + \frac{1}{k_n}$$

(g) Spring Combinations, Which are Partly in Parallel and Partly in Series

Obtain equivalent spring constants for each set of parallel or series springs separately and then combine. For example, in the sketch shown on the left, the springs k_1 and k_2 are equivalent to a single spring, the spring constant of which, k_{e1}, is given by:



$$\frac{1}{k_{e1}} = \frac{1}{k_1} + \frac{1}{k_2} = \frac{k_1 + k_2}{k_1 k_2}$$

or

$$k_{e1} = \frac{k_1 k_2}{k_1 + k_2}$$

The three springs, k_3, k_4, k_5 in parallel are equivalent to a single spring, the spring constant, k_{e2}, of which, is given by

$$k_{e2} = k_3 + k_4 + k_5$$

Now equivalent springs k_{e1} and k_{e2} are in series. Hence, the spring constant, k, of the equivalent spring for the entire system, is given by

$$\frac{1}{k} = \frac{1}{k_{e1}} + \frac{1}{k_{e2}}$$

$$\therefore k = \frac{k_{e1} k_{e2}}{k_{e1} + k_{e2}} = \frac{\left(\frac{k_1 k_2}{k_1 + k_2}\right)(k_3 + k_4 + k_5)}{\frac{k_1 k_2}{k_1 + k_2} + (k_3 + k_4 + k_5)}$$

$$\therefore k = \frac{(k_1 k_2)(k_3 + k_4 + k_5)}{k_1 k_2 + (k_1 + k_2)(k_3 + k_4 + k_5)}$$

(h) Geared Systems

In such systems, the system compliance and inertia is often referred to one shaft, usually the motor shaft, or drive shaft. The figure below shows a motor of inertia l_M driving a load of inertia l_L, through two gear-pinion reductions (Inertias l_{p1}, l_{g2}, l_{p2}, l_{g3}). The torsional compliance of the three shafts 1,2,3 of negligible inertia are k_1, k_2, k_3 respectively. Referred to the motor shaft, the overall compliance, k, of the system and the equivalent moment of inertia, l, of the system (also called the reflected moment of inertia) are determined as follows:

The equivalent spring constant, k, is given by:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{n_{21}^2 k_2} + \frac{1}{n_{31}^2 k_3}$$

where n_{21}, n_{31} are the gear ratios of shafts 2, 3 to shaft 1 respectively. The moment of inertia, l, of the system referred to the motor shaft (shaft #1) is given by:

$$l = (l_M + l_{p1}) + n_{21}^2 (l_{g2} + l_{p2}) + n_{31}^2 (l_{g3} + l_L)$$